METHODS OF RATIONALIZATION OF DENOMINATORS OF POLYNOMIALS WITH IRRATIONAL ROOTS: APPLICATION OF SYLVESTER MATRIX AND RESULTANT

Manasova Ayjan
Sh.Rashidov Samarkand State University

Annotation: *This study explores advanced techniques for rationalizing polynomial denominators with irrational roots through the application of Sylvester matrices and resultants. The proposed methods aim to simplify complex mathematical expressions, contributing to improved computational precision in diverse fields such as physics, engineering, and theoretical computer science.*

Keywords: *denominator rationalization, irrational roots, Sylvester matrix, resultant, Bezu identity, and polynomials.*

Questions related to the rationalization of denominators arose at the early stages of the development of algebra, when researchers faced the need to simplify mathematical expressions with irrational numbers. Historically, the problem of irrationality first appeared in the works of ancient Greek mathematicians, in particular Pythagoras and his followers, who detected irrational numbers in the study of geometric ratios. However, a more systematic study of denominator rationalization began later, with the development of polynomial algebra in the works of such scientists as Rene Descartes and Isaac Newton.

In the 19th century, mathematics received a major boost in the development of the theory of polynomials and rational fractions with the introduction of the Sylvester matrix and the resultant -- tools that made it possible to efficiently determine the greatest common divisor(GCD) of polynomials and simplify rational fractions with irrational components. These methods proved to be extremely useful in algebraic calculations, as they avoided complex expressions with irrationalities. expressions with irrationals in the denominator, which was an important advance for the accuracy and simplification of mathematical models.

In modern mathematical research and computational algebra. the rationalization of denominators remains a pressing issue. Complex mathematical models in physics, engineering and theoretical computer science often contain expressions that need to be simplified for a more accurate numerical analyses. Methods based on the use of the Sylvester matrix and the resultant, allow not only to get rid of the irrationality in denominators, but also offer efficient algorithms for program simplification of polynomial expressions.

In this context, denominator rationalization problems become a an integral part of algebraic analysis and require the development of methods that maximize the simplicity and accuracy of the expressions. expressions. One such method is the use of the Sylvester matrix and the resultant to obtain rational polynomials that preserve the basic properties of the original expressions. Let's take a closer look in detail the rationalization method using these tools.

Let $f(x), g(x), g_1(x)$ be polynomials from $A[x]$, γ_n are roots of $f(x)$. And let the rational fraction $\frac{g_1(x)}{g(x)}$ be given . We will consider the case when the coefficients of polynomials $f(x), g(x), g_1(x)$ are rational. However, among the roots $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma\_n$ of the polynomial $f(x)$ there are irrational ones.

In this case, we can set the problem of eliminating the irrationality in the denominator of the expression $\frac{g_1(\gamma)}{g(\gamma)}$.

Here $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n\}$

Theorem[1]. If $R(f, g) \neq 0$, then there exists a polynomial $G(x)$ that solves t the question of eliminating irrationality. Provided $\deg(G) < n$. Such a polynomial is uniquely defined. Its coefficients will depend rationally on the coefficients f(x),g(x),g_1 (x).In this context, the set γ is defined as follows: $\gamma = \{\gamma_1, \gamma_{2,} \gamma_{3,\cdots,}\gamma_n\}$

Theorem 1. Let $R(f, g) \neq 0$. Then, there exists a polynomial $G(x)$, of degree less than n that resolves the issue of denominator irrationality. The coefficients of $G(x)$ are rational and depend explicitly on the coefficients of the polynomials $f(x)$, $g(x)$, and $g_1(x)$.

Proof. It is straightforward to verify that there exists a polynomial $u(x)$ that satisfies the Bezu identity:

$$v(x)f(x) + u(x)g(x) = 1,$$

This, in fact, determines the solution to the problem. On an arbitrary root λ_j of the polynomial $f(x)$, this identity takes the form $\frac{1}{g(\lambda_j)} = u(\lambda_j)$. Consequently, the polynomial

$$G(x) = g_1(x)u(x)$$

This is the desired polynomial. As the coefficients of $u(x)$ are expressed in terms of the coefficients of the polynomials $f(x)$ and $g(x)$, it follows that the coefficients of the polynomial $G(x)$ will be rational numbers. Moreover, any polynomial of the form $G(x) + q(x)f(x)$, where $q(x) \in \mathbb{Q}[x]$, can be a solution to the problem. In particular, we can take $G(x)$ to be the residue resulting from dividing $g_1(x)u(x)$ by $f(x)$. This solution is unique if we impose the condition stated in the theorem.

The matrix M is referred to as the Sylvester matrix for the polynomials $f(x) = a_0x^n + a_0x^{n-1} + \cdots + a_n$ and $g(x) = b_0x^m + b_0x^{m-1} + \cdots + b_m$. The order of the matrix M is $m + n$. Its structure is as follows:

The Sylvester matrix M is defined as follows:

$$M = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{n-1} & a_n & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & & & \ddots & \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & & \cdots & a_{n-1} & a_n \\ 0 & 0 & \cdots & & b_0 & b_1 & \cdots & \cdots & b_{m-1} & b_m \\ 0 & 0 & \cdots & b_0 & b_1 & & & \cdots & b_m & 0 \\ \vdots & & \iddots & \cdots & & & & & & \vdots \\ 0 & b_0 & \cdots & \cdots & b_m & 0 & \cdots & & \cdots & 0 \\ b_0 & \cdots & \cdots & b_m & 0 & \cdots & & & & 0 \end{pmatrix}$$

The determinant of the matrix M is referred to as the resultant of the polynomials $f(x)$ and $g(x)$ (see [1]).

$$f(x)v(x) + g(x)u(x) \equiv R(f,g) \qquad (*)$$

The left-hand side is referred to as the linear representation of the greatest common divisor (see [2]).

$$f(x)v(x) + g(x)u(x) \equiv GCD(f,g),$$

It can be seen that $\mathrm{GCD}(f,g)$ and $R(f,g)$ are related.

The polynomials $u(x)$ and $v(x)$ can be found by replacing the last column of the resultant polynomials $f(x)$ and $g(x)$ by the vectors $(0,0,\cdots,x^{n-2},x^{n-1})$ and $(x^{m-1},x^{m-2},\cdots,0,0)$ respectively.

Upon substituting $x = \gamma$ into the identity $(*)$.

We obtain: $g(\gamma)u(\gamma) = R(f,g)$

This expression can be expressed in the following form:

$$\frac{u(\gamma)}{R(f,g)} = \frac{1}{g(\gamma)}$$

The following polynomial can be taken as the polynomial $G(x)$ to be sought:

$$G(x) = \frac{g_1(\gamma)u(\gamma)}{R(f,g)} \quad (**)$$

For example, one may choose to eliminate the irrationality present in the denominator of the given expression.

The expression in question is $\frac{\gamma}{\gamma^3-1}$, where γ represents the root of the polynomial x^5-4x-2.

Solution: In order to find $u(x)$ and $v(x)$ for the polynomials $f(x) = x^5 - 4x - 2$ and $g(x) = x^3 - 1$, it is necessary to use the identity $(*)$.

$$R(f,g) = \begin{vmatrix} 1 & 0 & 0 & 0 & -4 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -4 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{vmatrix} = 95$$

In lieu of the last column of the determinant, the vectors $(0,0,0,1,x,x^2,x^3,x^4)$ and $(x^2,x,1,0,0,0,0,0)$ are substituted.

Thus, we obtain the result $u(x) = 18x^4 - 7x^3 + 8x^2 + 18x - 79$ and $v(x) = -18x^2 + 7x - 8$.

The formula $(**)$ is now employed to calculate $G(x)$:

$$G(x) = \frac{18x^5 - 7x^4 + 8x^3 + 18x^2 - 79x}{95}$$

The remainder obtained by dividing $G(x)$ by $f(x)$ is as follows:

$$G_1(x) = \frac{-7x^4 + 8x^3 + 18x^2 - 7x + 36}{95}$$

This solution is also the only one among polynomials of powers less than 5.

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