THEORETICAL FOUNDATIONS AND PRACTICAL APPLICATIONS OF BÉZOUT'S THEOREM IN ALGEBRA AND PARAMETRIC POLYNOMIAL ANALYSIS

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Annotation: *This thesis explores Bézout's theorem, which connects polynomial division and root identification, emphasizing its theoretical foundations and historical significance through François Bézout's contributions. Practical examples demonstrate the theorem's applications in solving equations, analyzing parametric polynomials, and optimizing parameters in research. The relevance of Bézout's theorem is highlighted in modern studies of algebraic structures and its applications in science and engineering.*

Keywords: *Bézout's theorem, polynomial division, roots of polynomials, algebraic equations, François Bézout, parametric polynomials, root multiplicity, applied mathematics, system optimization.*

Definition of the problem and relevance

Bézout's theorem is one of the key theorems of algebra because it describes the relationship between dividing polynomials and finding their roots. In particular, it helps to determine whether some number is a root of a given polynomial, and shows that if the number c is a root of the polynomial $f(x)$, then the bipartition $(x - c)$ divides $f(x)$ without remainder. This statement can be expressed as a decomposition:

$$\mathrm{f(x) = (x - c) \cdot q(x) + r}$$

where $q(x)$ is the quotient , r is the remainder. 2

The important point of the theorem is that if $r = 0$, then the number c is the root of $f(x)$, which means $f(c) = 0$.

This theorem also has practical applications in solving algebraic equations, systems of equations, finding roots (multiples among others), and in polynomial factorization

Purpose of work

The main aims of this thesis are: to justify the importance of Bézout's theorem, practical applications, and a historical and theoretical overview of François Bézout's contribution to the development of algebra and the study of his theorem in mathematics.

Definition and formulation of the theorem

Bézout's Theorem

A number $c \in K$ is a root of a polynomial $f(x) \in K[x]$ if and only if $f(x) \vdots (x - c)$ without remainder. 2

Historical background

François Bézout (1736 - 1783) was a French mathematician who made significant contributions to algebra and number theory. His work influenced the further development of mathematical analysis and algebra, and has applications in modern mathematics, especially in field theory and algebraic geometry.

One of François Bézout's most significant results is his polynomial division theorem, better known as Bézout's root theorem. Bézout's theorem came about as a consequence of his studies of the algebraic structure of polynomials, especially their decomposition into simpler components (multipliers).

Bézout's key contribution lies in his peculiarity of presenting this in analytical form, which allowed mathematicians of subsequent generations to use his result in various applications, including: factorization of polynomials, finding roots, and developing methods for numerical solution of equations.

Proof of Bézout's theorem

The polynomial $f(x) \vdots (x-c)$ with remainder r, which can be written as:

$$f(x) = (x-c) \cdot q(x) + r$$

where $q(x)$ is the quotient, r is the remainder . When $x = c$ it will be of the form:

$$f(c) = (c-c) \cdot q(c) + r$$
$$f(c) = 0 \cdot q(c) + r = r$$

Hence, $r = f(c)$. If $f(c) = 0$, then $r = 0$ as well, indicating that $f(x) \vdots (x-c)$. [3]

One of the features of this theorem and its proof is its universality, more precisely, the proof applies to polynomials of any degree and quotient from any field (ring), also Bézout's theorem is appreciated for its accessibility and rigorous mathematical soundness.

Application of Bézout's theorem

Example 1. Consider a cubic equation of the form:

$$f(x) = \mathrm{x}^3 - 10\mathrm{x}^2 + 27\mathrm{x} - 18 = 0$$

To find the root of this polynomial, we start by substituting the possible root values. Let x_0 be the whole root of this equation, hence it must be a divisor of the free term (-18). All possible values of the root will be of the form:

$$\mathrm{x}_0 = \pm 1; \pm 2; \pm 3; \pm 6; \pm 9; \pm 18$$

Let now $\mathrm{x}_0 = \pm 1 \Rightarrow f(-1) = -1 - 10 - 27 - 18 \neq 0$ and $f(1) = 1 - 10 + 27 - 18 = 0$

By theorem, we have that if $f(\mathrm{x}_0) = 0$, then $\mathrm{x} - \mathrm{x}_0$ is a divisor of $f(x)$, that is $\Rightarrow f(x) \vdots (x-1)$ [4]

$x^3 - 10x^2 + 27x - 18 \quad | \quad \mathrm{x} - 1$

$-\ x^3 - x^2 \quad | \quad \mathrm{x}^2 - 9\mathrm{x} + 18$

$-9\mathrm{x}^2 + 27\mathrm{x}$

$-\ -9\mathrm{x}^2 + 9\mathrm{x}$

$18\mathrm{x} - 18$

$-$

$18x - 18$

0

$\Rightarrow f(x) = (x-1)(x^2 - 9x + 18) = (x-1)(x-3)(x-6)$

Further without picking the roots, by Bézout's theorem we have that the given polynomial has 3 real, integer roots $(1; 3; 6)$.

Example 2. 5

Given a parametric polynomial of the form: $f(x) = 2x^4 + px^3 + 3x^2 - qx + 5$

Problem: At what values of p and q is the number $x_0 = 1$ a root of the polynomial $f(x) = 2x^4 + px^3 + 3x^2 - qx + 5$ of multiplicity greater than 1?

The number x_0 is a root of a polynomial $f(x)$ of multiplicity greater than 1, if and only if $f(x_0) = 0$ and $f`(x_0) = 0$

In our case, $f`(x) = 8x^3 + 3px^2 + 6x - q$

Let's make a system: $\begin{cases} f(1) = 0 \\ f`(1) = 0 \end{cases} \Rightarrow \begin{cases} p - q = 10 \\ 3p - q = 14 \end{cases} \Rightarrow \begin{cases} p = 2 \\ q = -8 \end{cases}$

Hence, when $p = 2$ $q = -8$, this polynomial has a root $(x_0 = 1)$ of multiplicity greater than 1.

Bezu's theorem states that if $x_0 = 1$, then $f(1) = 0$. This requirement gives one equation relating the parameters p and q. For $x = 1$ to be a root of multiplicity greater than 1, it is also necessary that the derivative of $f`(x)$ at $x = 1$ also converges to zero, i.e. $f`(1) = 0$. This condition leads to a second equation for the parameters p and q.

These conditions, expressed as a system of equations, made it possible to find specific values of the parameters at which the polynomial $f(x)$ has the desired form.

Conclusion: In our study we have considered the conditions under which a given number becomes a root of a polynomial with a certain multiplicity, and have shown how this can be used to analyze parametric polynomials. This approach is important not only in theoretical algebra, but also in applications that require the analysis of parametric polynomials or the study of the root structure of equations.

The application of Bézout's theorem to modern problems, such as finding optimal parameters in systems of equations, emphasizes its relevance. The theorem remains a powerful tool in the hands of the mathematician, providing clarity and structure in research and enabling the solution of a wide range of problems, from algebraic to applied problems.

Thus, the study and use of Bézout's theorem contributes to a deeper understanding of algebraic structures and allows practical problems in various fields of science and engineering to be solved more confidently and accurately.

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